# Variational theory for nematoacoustics

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(Received 20 May 2009; published 14 September 2009)

The effect of an ultrasonic wave on the nematic texture has long been known, but its interpretation in terms of a coherent dynamical theory has not yet been achieved. A proposal for such a theory is made in this paper. The diverse theoretical approaches attempted in the past to describe the interaction between sound and nematic molecular orientation are briefly summarized. A theory for second-grade fluids, which provides the appropriate theoretical background for nematoacoustics, is also revived. An explicit application of the proposed theory to a simple computable case is given, which yields predictions that are qualitatively confirmed by a number of experimental results.

DOI: 10.1103/PhysRevE.80.031705 PACS number(s): 61.30.−v, 62.60.+v

## I. INTRODUCTION

Experimental acoustic studies in nematic liquid crystals have a long history, including early contributions from pioneers of liquid crystal science such as Lehman and Zolina [1]. Several reviews provide accounts on the effect of an acoustic field on the orientation of nematic molecules; we only quote [1–3] among the most recent ones, which also report the still unappeased debate between the different theories that have attempted to explain the interaction between acoustic waves and nematic textures.

The main experimental findings that called for explanation were the anisotropy observed in both attenuation and speed of sound in the propagation of ultrasonic waves in nematic liquid crystals where the orientation of the director is kept fixed by an aligning magnetic field [4–8] and the reorientating action exerted on a uniformly aligned nematic cell by the propagation of ultrasonic waves in the absence of any other external action [9–11]. This evidence purported the hypothesis that a condensation wave can affect the director orientation in a way similar in its appearance, though not in its cause, to the action exerted by an external magnetic or electric field. Actually, the *acousto-optic* effect, as it is often called, produces an alteration of the birefringence in a nematic cell, which is easily detected and closely resembles the optic effect induced by an external field, as if the acoustic field could also impart a torque on the nematic director.

The theories so far proposed to explain the acoustic action on nematic liquid crystals can essentially be grouped in two wide categories: theories that explain the acoustic-nematic interaction by means of an intermediate hydrodynamic flow of a sort or another, and theories that explain the acoustic-nematic interaction through a direct coupling between acoustic field and nematic director, with its own associated elastic energy. The theories in the former category build essentially on the classical Ericksen-Leslie theory [12,13] and presume that an acoustic wave is capable of inducing a steady nonuniform flow which in turn acts on the director field, thus distorting it, whereas the theories in the latter category posit an elastic interaction between an acoustic wave and the director field, which is also capable of inducing distortions in the absence of any induced flow.

The major hydrodynamic mechanism that has been imagined to transmit torque from the acoustic field to the nematic director is a nonlinear coupling relying on the occurrence of a variant of Reynolds stresses in the fluid. Related to these stresses is also the notion of *acoustic streaming*, which describes a phenomenon also known for dissipative isotropic fluids; a rather general description of these concepts and the mathematical techniques connected to them can be found in [14] (see, in particular, Sec. 4.7). Here, following in part [15], we shall be contented with outlining the general ideas underlining this method, to the extent that it may be applied to our context. Another application of these ideas is illustrated in [16].

Let $u$ be any of the fields describing the flow: it may designate either the pressure or the density, a component of the velocity field or a component of the nematic director. We expand $u$ in the form

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + o(\varepsilon^2),$$

where $\varepsilon$ is a perturbation parameter, $u_0$ is the equilibrium value of $u$, and $u_1$ and $u_2$ are the first- and second-order corrections to $u_0$, respectively. In a plane-wave solution to the dynamical equations of the Ericksen-Leslie theory, $u_1$ has zero time average, whereas $u_2$ can in general be written as

$$u_2 = \bar{u}_2 + \hat{u}_2,$$

where it is decomposed in a steady component, $\bar{u}_2$, and a varying component, $\hat{u}_2$, oscillating at a frequency twice the frequency of $u_1$, which like this latter averages out to zero. The dynamical equations for the various fields like $\bar{u}_2$, which capture the slow, second-order evolution of the fluid, are derived by averaging in time the contributions to the general dynamical equations that are second order in $\varepsilon$, as is typical in any perturbation method. Such second-order equations will invariably be affected by the time averages of terms quadratic in $u_1$, which will thus act as forces for the growth of inhomogeneities in $\bar{u}_2$. This is the essence of the acoustic streaming method applied in [15] to the Ericksen-Leslie dynamic equations for nematic liquid crystals. The second-order character of the stresses responsible for the onset of the

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steady second-order flow makes them resemble Reynolds stresses of ordinary fluid dynamics (see, for example, pp. 328–330 of [14]). These stresses are responsible for making turbulent velocity perturbations about a mean flow interfere with the mean flow itself, thus generating sound. Conversely, waves propagating through a mean flow affect it through exactly the same mechanism (see p. 330 of [14]). This is indeed the conceptual connection between turbulence and acoustic streaming, also implied in the extension to nematic liquid crystals proposed by [15]. In summary, according to [15], acoustic streaming in nematic liquid crystals would be responsible for the hydrodynamic coupling that transfers torque from a traveling ultrasonic wave to the nematic director.

Essentially the same approach as in [15], though with some apparent variants, was more recently adopted in [17] and further applied in a series of other works [18–20] to explain nematic alignment produced by ultrasonic waves. Within a slightly different category, though still postulating a hydrodynamic mediation, falls the explanation of the acoustic action on the nematic director proposed in [21]. In general, sound is known to produce a *radiation pressure* in the medium where it propagates (see, for example, Sec. 64 of [22]). Such a pressure is to be distinguished from the acoustic pressure, often also called the *excess* pressure; the latter is the difference between the pressure carrying an acoustic wave and the uniform pressure of the unperturbed medium; it averages out to zero in time and so has no net mechanical effect. The radiation pressure is the time average of the second-order correction to the unperturbed pressure and it is determined by the second-order components of the dynamical equation. In isotropic fluids, the radiation pressure can only induce a force along the direction of propagation, but in anisotropic fluids, such as nematic liquid crystals, the time average of second-order stresses may also induce transverse actions resulting in a torque on the nematic director. As for the acoustic streaming, such a torque would thus be of a viscous nature.

Here we shall follow a conceptual avenue that essentially differs from those already outlined in the nature of the postulated aligning torque, which will be elastic rather than viscous. Thus no flow will be needed for an acoustic field to act upon the nematic director. This line of thought first arose in [23], whose experimental results suggested to supplement the elastic energy density with the following acoustic contribution

$$W_{\mathrm{a}} = c_1 k^2 + c_2(\boldsymbol{n}\cdot\boldsymbol{k})^2, \qquad (1)$$

where $c_1$ and $c_2$ are constitutive constants, $\boldsymbol{k}$ is the acoustic wave vector, and $\boldsymbol{n}$ is the nematic director. A similar interaction, even if not explicitly formulated as in Eq. (1), was also adduced in [11] to interpret some acousto-optical observations.

Dion and Jacob [24] are often credited with having first proposed a direct interaction between acoustic propagation and molecular alignment. However, the interpretation of this interaction within the general principle of minimum entropy production [25] has obscured its elastic character, thus bringing it into the realm of dissipation, where it does not really belong to. In Dion and Jacon's own words [24], "in a medium with acoustical anisotropy, the molecules tend to reorient so as to minimize propagation losses." Such an interpretation of the acoustic-molecular interaction has fueled controversies and caused misunderstanding (exemplary to this effect is the comment on Dion's work on p. 184 of [3]).

As proposed independently and almost simultaneously in [26,27], we hold that the acoustic-nematic interaction is of an elastic nature and results from the coupling between the density gradient induced by the acoustic wave and the average molecular orientation represented by the nematic director. Since the typical characteristic times of acoustic waves are much shorter than the director's relaxation time, it is actually the time-averaged interaction energy that will affect the nematic elastic energy. Both papers [26,27] were followed by further extensions of the original assumption along with the first experimental confirmations of that theory; in particular, we refer the reader to the series of works [28–33]. Here, we shall indeed posit a slight variant of this assumption and we shall interpret through the ensuing theory experiments long published in the literature, though never completely explained.

At the time scale of the acoustic oscillations, at which the director texture can be regarded as prescribed and immobile, a nematic liquid crystal behaves like an anisotropic *Korteweg fluid*, that is, like an elastic fluid whose free energy density also depends on the density gradient. Korteweg [34] first considered a special isotropic fluid with the elastic stress tensor depending on both the first and second gradients of the density field; he built his capillarity theory on such a constitutive assumption, as also recalled in [35] (see, in particular, pp. 513–515). Under appropriate assumptions, Korteweg stress tensor is *hyperelastic*, that is, it can be derived from a potential that depends on the density and its first gradient (see also Sec. 18 of [36]).

In the following section, we shall present a general variational theory for Korteweg fluids, which will be further adapted to nematoacoustics in Sec. III, where we show how the time-averaged elastic actions associated with acoustic propagation affect the dynamics of nematic liquid crystals. In particular, we shall draw the consequences of our general theory for the propagation of acoustic plane waves in a uniformly aligned nematic liquid crystal: we shall compute both the speed of propagation and the wave attenuation as functions of frequency, propagation direction, and nematic viscosities. In the closing Sec. IV, directions for future work are also indicated.

## II. KORTEWEG FLUIDS

In this section we consider a perfect second-grade fluid, whose elastic energy density is a function of both the mass density $\varrho$ and its spatial gradient $\nabla\varrho$. Our objective is identifying both stresses and traction laws relevant to this class of fluids. The extension to nematoacoustics of the balance laws derived here will be the object of Sec. III, where the time scale at which a nematic liquid crystal behaves like a Korteweg fluid will be separated from the time scale at which only the average effects of such a behavior survive.

### A. Principle of virtual power

Here, as in the classical treatment of second-grade materials of Toupin [37,38] (see also [39] for a more recent application of the same method), we start by deriving both balance equations and traction laws of statics from a principle of virtual power.

Let the internal energy $\mathcal{F}_K(\mathcal{P})$ of the fluid occupying the subbody $\mathcal{P}$ of the body $\mathcal{B}$ be given by

$$\mathcal{F}_K(\mathcal{P}) := \int_{\mathcal{P}} \varrho \sigma_K(\varrho, \nabla\varrho) dV, \tag{2}$$

where $\sigma_K$ is the internal energy per unit mass and $V$ denotes the volume measure. Let $\mathcal{W}^{(e)}(\mathcal{P})$ be the power expended in a virtual motion by the external actions exerted on $\mathcal{P}$. Following [38,39], we posit for $\mathcal{W}^{(e)}(\mathcal{P})$ the following form,

$$\mathcal{W}^{(e)}(\mathcal{P}) := \int_{\mathcal{P}} \boldsymbol{b} \cdot \boldsymbol{v} dV + \int_{\partial\mathcal{P}} \left( \boldsymbol{t} \cdot \boldsymbol{v} + \boldsymbol{m} \cdot \frac{\partial \boldsymbol{v}}{\partial \nu} \right) dA, \tag{3}$$

where $A$ is the area measure and $\boldsymbol{v}$ is the velocity field inducing a *virtual* flow of the subbody $\mathcal{P}$, thought of as *carved* out of the whole body $\mathcal{B}$, while the actions exerted both in its bulk and on its boundary are held fixed, and the subbody $\mathcal{B}\backslash\mathcal{P}$ surrounding it is equally *frozen*.

In Eq. (3), $\boldsymbol{b}$ is the external *body force* defined in the whole of $\mathcal{B}$, while $\boldsymbol{t}$ and $\boldsymbol{m}$ are surface *contact* actions, the former expending power against $\boldsymbol{v}$, and so identifiable as a force, the latter expending power against the normal derivative of $\boldsymbol{v}$,

$$\frac{\partial \boldsymbol{v}}{\partial \nu} := (\nabla \boldsymbol{v})\boldsymbol{\nu}. \tag{4}$$

The unit vector $\boldsymbol{\nu}$ is the outer normal to $\partial\mathcal{P}$ and , according to Toupin [38], $\boldsymbol{m}$ is identifiable as a *hypertraction*. The hypertraction $\boldsymbol{m}$ would not be present in a classical simple fluid, for which the elastic energy density is independent of $\nabla\varrho$; as is soon to be shown, its presence in Eq. (3) is needed to counterbalance the internal power associated with the dependence of $\sigma_K$ on $\nabla\varrho$. While the body force $\boldsymbol{b}$ is a prescribed source, both surface actions $\boldsymbol{t}$ and $\boldsymbol{m}$ should be considered as unknown functionals of the boundary $\partial\mathcal{P}$ to be determined so as to comply with the variational principle posited by the theory. For statics, this principle is illustrated below; it is intended to provide both the balance equations valid within the body at equilibrium and the traction laws revealing how contact actions are transmitted through the boundary of internal subbodies.

We shall require the equilibrium configurations of the body $\mathcal{B}$ to be such that, for every subbody $\mathcal{P} \subset \mathcal{B}$,

$$\dot{\mathcal{F}}_K(\hat{\mathcal{P}}(t))|_{t=0} = \mathcal{W}^{(e)}(\mathcal{P}), \tag{5}$$

where the time derivative of $\mathcal{F}_K$ is meant to be computed along a virtual incipient flow $\hat{\mathcal{P}}(t)$ of $\mathcal{P}$.

A virtual flow $\hat{\mathcal{P}}(t)$ of $\mathcal{P}$ is described by a velocity field $\boldsymbol{v}(\cdot,t)$ defined for every $t \in [0,T]$ with $T > 0$ on the evolved subbody $\hat{\mathcal{P}}(t)$, that is, on the union of all positions attained at time $t$ by the points constituting $\mathcal{P}$ at time $t=0$. Formally, for every $t \in [0,T]$, $p(t) \in \hat{\mathcal{P}}(t)$ whenever the trajectory $t \mapsto p(t)$ solves the evolution problem

$$\dot{p}(t) = \boldsymbol{v}(p(t),t), \text{ with } p(0) \in \mathcal{P}, \tag{6}$$

so that $\hat{\mathcal{P}}(0) = \mathcal{P}$.

### B. Korteweg stress

The time derivative of $\mathcal{F}_K$ in Eq. (5) is to be computed with the aid of Reynold's transport theorem in the Eulerian formalism, which we now recall from p. 105 of [40]. For a functional $\Phi$ defined on the evolving subbody $\hat{\mathcal{P}}(t)$ as

$$\Phi(\hat{\mathcal{P}}(t)) := \int_{\hat{\mathcal{P}}(t)} \varphi(x,t) dV(x), \tag{7}$$

where $\varphi(\cdot,t)$ is a smooth scalar field on $\hat{\mathcal{P}}(t)$, Reynold's transport theorem says that

$$\dot{\Phi}(\hat{\mathcal{P}}(t)) = \int_{\hat{\mathcal{P}}(t)} (\varphi \operatorname{div} \boldsymbol{v} + \dot{\varphi}) dV, \tag{8}$$

where $\dot{\varphi}$ is the *material* time derivative of $\varphi$, that is, the derivative of $\varphi$ computed along the trajectories in Eq. (6),

$$\dot{\varphi} := \frac{d}{dt}\varphi(p(t),t) = \nabla\varphi \cdot \boldsymbol{v} + \frac{\partial \varphi}{\partial t}, \tag{9}$$

where the gradient $\nabla$ insists on the spatial variable only.

A mass evolution is associated with the virtual flow $\boldsymbol{v}$; it is described by a mass density function $\varrho(\cdot,t)$ defined on $\hat{\mathcal{P}}(t)$ for every $t \in [0,T]$. In particular, the functional

$$M(\hat{\mathcal{P}}(t)) := \int_{\hat{\mathcal{P}}(t)} \varrho(x,t) dV(x),$$

which represents the mass stored in $\hat{\mathcal{P}}(t)$, is a special form of $\Phi$ in Eq. (7). By Eq. (8), requiring

$$\dot{M}(\hat{\mathcal{P}}(t)) \equiv 0 \text{ for all } \mathcal{P} \subset \mathcal{B},$$

which translates the conservation of mass along the virtual motion of any subbody, is equivalent to the continuity equation

$$\dot{\varrho} + \varrho \operatorname{div} \boldsymbol{v} = 0, \tag{10}$$

which must hold identically along all virtual motions of $\mathcal{P}$.

Equation (10) gives $\dot{\mathcal{F}}_K$ a simpler form. By applying Eq. (8) to $\mathcal{F}_K$ in Eq. (2), we indeed obtain that

$$\begin{aligned} \dot{\mathcal{F}}_K(\hat{\mathcal{P}}(t)) &= \int_{\hat{\mathcal{P}}(t)} (\varrho\sigma_K \operatorname{div} \boldsymbol{v} + \dot{\varrho}\sigma_K + \varrho\dot{\sigma}_K) dV \\ &= \int_{\hat{\mathcal{P}}(t)} \varrho \dot{\sigma}_K dV, \end{aligned} \tag{11}$$

where, by the chain rule,

$$\dot{\sigma}_K = \frac{\partial \sigma_K}{\partial \varrho}\dot{\varrho} + \frac{\partial \sigma_K}{\partial \nabla\varrho}(\nabla\varrho)^{\cdot}. \tag{12}$$

Applying Eq. (9) to the vector field $\nabla\varrho$, we readily arrive at

$$(\nabla\varrho)^{\cdot} = (\nabla^2\varrho)\boldsymbol{v} + \frac{\partial}{\partial t}(\nabla\varrho). \tag{13}$$

Under the assumption that $\varrho$ be a sufficiently smooth function, also by Eq. (10), we see that

$$\begin{aligned}\frac{\partial}{\partial t}(\nabla\varrho) &= \nabla\left(\frac{\partial\varrho}{\partial t}\right)\\ &= -\nabla(\nabla\varrho\cdot\boldsymbol{v}) - \nabla(\varrho\,\mathrm{div}\,\boldsymbol{v})\\ &= -(\nabla^2\varrho)\boldsymbol{v} - (\nabla\boldsymbol{v})^{\mathsf{T}}\nabla\varrho - \nabla(\varrho\,\mathrm{div}\,\boldsymbol{v}),\end{aligned}$$

where the superscript $^{\mathsf{T}}$ denotes transposition, and thus Eq. (13) becomes

$$(\nabla\varrho)^{\cdot} = -\nabla(\varrho\,\mathrm{div}\,\boldsymbol{v}) - (\nabla\boldsymbol{v})^{\mathsf{T}}(\nabla\varrho).$$

By this latter equation, using Eq. (10), from Eqs. (11) and (12) we finally arrive at

$$\dot{\mathcal{F}}_{\mathrm{K}}(\hat{\mathcal{P}}(t))|_{t=0} = -\int_{\mathcal{P}}\varrho\left\{\frac{\partial\sigma_{\mathrm{K}}}{\partial\varrho}\varrho\,\mathrm{div}\,\boldsymbol{v} + \frac{\partial\sigma_{\mathrm{K}}}{\partial\nabla\varrho}\cdot[\nabla(\varrho\,\mathrm{div}\,\boldsymbol{v}) + (\nabla\boldsymbol{v})^{\mathsf{T}}\nabla\varrho]\right\}dV \tag{14}$$

since $\hat{\mathcal{P}}(0)=\mathcal{P}$. Integrations by parts and repeated use of the divergence theorem allow us to give Eq. (14) the following form,

$$\dot{\mathcal{F}}_{\mathrm{K}}(\hat{\mathcal{P}}(t))|_{t=0} = -\int_{\mathcal{P}}\mathrm{div}\,\mathbf{T}_{\mathrm{K}}\cdot\boldsymbol{v}dV + \int_{\partial\mathcal{P}}\mathbf{T}_{\mathrm{K}}\boldsymbol{\nu}\cdot\boldsymbol{v}dA - \int_{\partial\mathcal{P}}\varrho^2\frac{\partial\sigma_{\mathrm{K}}}{\partial\nabla\varrho}\cdot\boldsymbol{\nu}\,\mathrm{div}\,\boldsymbol{v}dA, \tag{15}$$

where

$$\mathbf{T}_{\mathrm{K}} := -p_{\mathrm{K}}\mathbf{I} - \varrho\nabla\varrho\otimes\frac{\partial\sigma_{\mathrm{K}}}{\partial\nabla\varrho} \tag{16}$$

is the *Korteweg stress tensor* and

$$p_{\mathrm{K}} := \varrho^2\frac{\partial\sigma_{\mathrm{K}}}{\partial\varrho} - \varrho\,\mathrm{div}\left(\varrho\frac{\partial\sigma_{\mathrm{K}}}{\partial\nabla\varrho}\right) \tag{17}$$

is the associated *Korteweg pressure*.

### C. Surface calculus

The second surface integral in Eq. (15) needs to be further transformed to give Eq. (15) a form compatible with Eq. (3). To this end, we recall from Sec. 2.3.6 of [41] the surface-divergence theorem.

Let $\mathcal{S}$ be a smooth, orientable, closed surface in the three-dimensional Euclidean space $\mathcal{E}$ and let $\boldsymbol{u}$ be a differentiable vector field on $\mathcal{S}$. The *surface divergence* of $\boldsymbol{u}$ is defined by

$$\mathrm{div}_{\mathrm{s}}\,\boldsymbol{u} := \mathrm{tr}\,\nabla_{\mathrm{s}}\boldsymbol{u},$$

where $\nabla_{\mathrm{s}}\boldsymbol{u}$ is the *surface gradient* of $\boldsymbol{u}$.

It can be shown that

$$\nabla_{\mathrm{s}}\boldsymbol{u} = (\nabla\hat{\boldsymbol{u}})\mathbf{P}(\boldsymbol{\nu}), \tag{18}$$

where

$$\mathbf{P}(\boldsymbol{\nu}) := \mathbf{I} - \boldsymbol{\nu}\otimes\boldsymbol{\nu} \tag{19}$$

is the projection onto the plane orthogonal to a unit normal $\boldsymbol{\nu}$ to $\mathcal{S}$, and $\hat{\boldsymbol{u}}$ is any smooth extension of $\boldsymbol{u}$ to a three-dimensional neighborhood of $\mathcal{S}$. It follows from Eqs. (18) and (19) that

$$\nabla_{\mathrm{s}}\boldsymbol{u} = \nabla\hat{\boldsymbol{u}} - (\nabla\hat{\boldsymbol{u}})\boldsymbol{\nu}\otimes\boldsymbol{\nu} = \nabla\hat{\boldsymbol{u}} - \frac{\partial\hat{\boldsymbol{u}}}{\partial\nu}\otimes\boldsymbol{\nu},$$

which, letting

$$\nabla_{\nu}\hat{\boldsymbol{u}} := \frac{\partial\hat{\boldsymbol{u}}}{\partial\nu}\otimes\boldsymbol{\nu}$$

and noting that $\nabla_{\mathrm{s}}\boldsymbol{u}=\nabla_{\mathrm{s}}\hat{\boldsymbol{u}}$, we can also rewrite as

$$\nabla\hat{\boldsymbol{u}} = \nabla_{\mathrm{s}}\hat{\boldsymbol{u}} + \nabla_{\nu}\hat{\boldsymbol{u}}, \tag{20}$$

whence we interpret $\nabla_{\nu}\hat{\boldsymbol{u}}$ as the *normal gradient* of $\hat{\boldsymbol{u}}$. By computing the trace of the tensors on both sides of Eq. (20), we conclude that

$$\mathrm{div}\,\hat{\boldsymbol{u}} = \mathrm{div}_{\mathrm{s}}\,\hat{\boldsymbol{u}} + \mathrm{div}_{\nu}\,\hat{\boldsymbol{u}}, \tag{21}$$

where

$$\mathrm{div}_{\boldsymbol{\nu}}\,\hat{\boldsymbol{u}} := \mathrm{tr}\,\nabla_{\boldsymbol{\nu}}\hat{\boldsymbol{u}} = \frac{\partial\hat{\boldsymbol{u}}}{\partial\nu}\cdot\boldsymbol{\nu} \tag{22}$$

is the *normal divergence* of $\hat{\boldsymbol{u}}$.

The surface-divergence theorem states that

$$\int_{\mathcal{S}}\mathrm{div}_{\mathrm{s}}\,\boldsymbol{u}dA = \int_{\mathcal{S}}\boldsymbol{u}\cdot\boldsymbol{\nu}\,\mathrm{div}_{\mathrm{s}}\,\boldsymbol{\nu}dA, \tag{23}$$

for all smooth vector fields $\boldsymbol{u}$ on $\mathcal{S}$. In Eq. (23), $\mathrm{div}_{\mathrm{s}}\,\boldsymbol{\nu}$ embodies the differential properties of the surface $\mathcal{S}$,

$$\mathrm{div}_{\mathrm{s}}\,\boldsymbol{\nu} = \mathrm{tr}\,\nabla_{\mathrm{s}}\boldsymbol{\nu} = 2H, \tag{24}$$

where $\nabla_{\mathrm{s}}\boldsymbol{\nu}$ is the *curvature tensor*, which enjoys the properties

$$(\nabla_{\mathrm{s}}\boldsymbol{\nu})^{\mathsf{T}} = \nabla_{\mathrm{s}}\boldsymbol{\nu} \quad\text{and}\quad (\nabla_{\mathrm{s}}\boldsymbol{\nu})\boldsymbol{\nu} \equiv \mathbf{0},$$

and $H$ is the *mean curvature* of $\mathcal{S}$.

Similarly, for a smooth scalar field $\chi$ on $\mathcal{S}$, the surface gradient-integral theorem says that

$$\int_{\mathcal{S}}\nabla_{\mathrm{s}}\chi dA = \int_{\mathcal{S}}\chi(\mathrm{div}_{\mathrm{s}}\,\boldsymbol{\nu})\boldsymbol{\nu}dA. \tag{25}$$

### D. Traction and hypertraction

By Eqs. (21) and (22), we have that

$$\int_{\partial\mathcal{P}} \varrho^2 \frac{\partial \sigma_{\rm K}}{\partial \nabla \varrho} \cdot \boldsymbol{\nu} \,\mathrm{div}\, \boldsymbol{v} dA = \int_{\partial\mathcal{P}} \varrho^2 \frac{\partial \sigma_{\rm K}}{\partial \nabla \varrho} \cdot \boldsymbol{\nu}\left(\mathrm{div}_{\rm s}\, \boldsymbol{v} + \frac{\partial \boldsymbol{v}}{\partial \nu} \cdot \boldsymbol{\nu}\right) dA, \qquad (26)$$

and, using the identity

$$\varrho^2 \frac{\partial \sigma_{\rm K}}{\partial \nabla \varrho} \cdot \boldsymbol{\nu}\, \mathrm{div}_{\rm s}\, \boldsymbol{v} = \mathrm{div}_{\rm s}\left[\left(\varrho^2 \frac{\partial \sigma_{\rm K}}{\partial \nabla \varrho} \cdot \boldsymbol{\nu}\right)\boldsymbol{v}\right] - \nabla_{\rm s}\left(\varrho^2 \frac{\partial \sigma_{\rm K}}{\partial \nabla \varrho}\right) \cdot \boldsymbol{v}$$

and the surface-divergence theorem in Eq. (23), we can also write

$$\int_{\partial\mathcal{P}} \varrho^2 \frac{\partial \sigma_{\rm K}}{\partial \nabla \varrho} \cdot \boldsymbol{\nu}\, \mathrm{div}_{\rm s}\, \boldsymbol{v} dA = \int_{\partial\mathcal{P}} \left(\varrho^2 \frac{\partial \sigma_{\rm K}}{\partial \nabla \varrho} \cdot \boldsymbol{\nu}\right) 2H(\boldsymbol{v}\cdot\boldsymbol{\nu}) dA - \int_{\partial\mathcal{P}} \nabla_{\rm s}\left(\varrho^2 \frac{\partial \sigma_{\rm K}}{\partial \nabla \varrho}\right) \cdot \boldsymbol{v} dA.$$

Making use of both this equation and Eq. (26), we finally arrive at

$$\dot{\mathcal{F}}_{\rm K}(\hat{\mathcal{P}}(t))|_{t=0} = -\int_{\mathcal{P}} \mathrm{div}\, \mathbf{T}_{\rm K} \cdot \boldsymbol{v} dA - \int_{\partial\mathcal{P}} \varrho^2 \left(\frac{\partial \sigma_{\rm K}}{\partial \nabla \varrho} \cdot \boldsymbol{\nu}\right) \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{\nu}} \cdot \boldsymbol{\nu} dA + \int_{\partial\mathcal{P}} \left[\mathbf{T}_{\rm K}\boldsymbol{\nu} + \nabla_{\rm s}\left(\varrho^2 \frac{\partial \sigma_{\rm K}}{\partial \nabla \varrho} \cdot \boldsymbol{\nu}\right) - \left(\varrho^2 \frac{\partial \sigma_{\rm K}}{\partial \nabla \varrho} \cdot \boldsymbol{\nu}\right) 2H\boldsymbol{\nu}\right] \cdot \boldsymbol{v} dA. \qquad (27)$$

Inserting both Eqs. (27) and (3) into Eq. (5), and requiring the latter to be valid for every virtual flow $\boldsymbol{v}$ of $\mathcal{P}$ and for every subbody $\mathcal{P}$ of $\mathcal{B}$, we derive the equation

$$\boldsymbol{b} + \mathrm{div}\, \mathbf{T}_{\rm K} = \mathbf{0}, \qquad (28)$$

expressing the balance of external and internal forces at equilibrium in $\mathcal{B}$, and the traction laws

$$\boldsymbol{t} = \mathbf{T}_{\rm K}\boldsymbol{\nu} + \nabla_{\rm s}\left(\varrho^2 \frac{\partial \sigma_{\rm K}}{\partial \nabla \varrho} \cdot \boldsymbol{\nu}\right) - \varrho^2\left(\frac{\partial \sigma_{\rm K}}{\partial \nabla \varrho} \cdot \boldsymbol{\nu}\right) 2H\boldsymbol{\nu} \qquad (29)$$

and

$$\boldsymbol{m} = -\varrho^2\left(\frac{\partial \sigma_{\rm K}}{\partial \nabla \varrho} \cdot \boldsymbol{\nu}\right)\boldsymbol{\nu}, \qquad (30)$$

valid on the boundary $\partial\mathcal{P}$ of every subbody $\mathcal{P}$ of $\mathcal{B}$.

Equation (29) illustrates a notable variance from the linear dependence of the traction $\boldsymbol{t}$ onto the outer unit normal $\boldsymbol{\nu}$ established by Cauchy's classical theorem (see, for example, pp. 174–177 of [40]), a deviation typical of second-grade fluids. It should be noted, however, that by Eq. (24) $\boldsymbol{t}$ is still an odd function of $\boldsymbol{\nu}$, thus complying with Newton's action and reaction principle (see also p. 164 of [40]).

Equation (30) represents the hypertraction $\boldsymbol{m}$ as a function of $\boldsymbol{\nu}$; unlike $\boldsymbol{t}$, $\boldsymbol{m}$ is even in $\boldsymbol{\nu}$; it is delivered by a third-rank tensor $\mathsf{M}$, which Toupin [37,38] suggested to call a *hyperstress*: in Cartesian components,

$$(m_K)_i = M_{jik}\nu_j\nu_k,$$

with repeated indices implying summation and

$$M_{jik} := -\varrho^2 \frac{\partial \sigma_{\rm K}}{\partial \varrho_{,j}} \delta_{ik}, \qquad (31)$$

where a comma denotes differentiation with respect to Cartesian coordinates $(x_1, x_2, x_3)$ and $\delta_{ik}$ is Kronecker's symbol.

A property of the Korteweg stress in Eq. (16) is worth mentioning: it is necessarily symmetric. This property follows from a variant of the principle of frame indifference, that is, from the requirement that the free energy time rate $\dot{\mathcal{F}}_{\rm K}(\hat{\mathcal{P}}(t))|_{t=0}$ be zero for every subbody $\mathcal{P} \subset \mathcal{B}$ along any rigid motion. To prove this, we begin by representing a rigid motion through the flow

$$\boldsymbol{v}_{\rm R}(x) = \boldsymbol{v}_{\rm R}(o) + \mathbf{W}\boldsymbol{x}, \qquad (32)$$

where $\mathbf{W}$ is a skew tensor, also called the *spin tensor* of $\boldsymbol{v}_{\rm R}$ and $\boldsymbol{x} := x - o$. It readily follows from Eq. (32) that $\nabla \boldsymbol{v}_{\rm R} = \mathbf{W}$, and so for a rigid motion div $\boldsymbol{v}_{\rm R} = 0$. Thus Eq. (14) becomes

$$\dot{\mathcal{F}}_{\rm K}(\hat{\mathcal{P}}(t))|_{t=0} = \int_{\mathcal{P}} \varrho \frac{\partial \sigma_{\rm K}}{\partial \nabla \varrho} \cdot \mathbf{W}\, \nabla \varrho dV = -\mathbf{W} \cdot \int_{\mathcal{P}} \varrho\, \nabla \varrho \otimes \frac{\partial \sigma_{\rm K}}{\partial \nabla \varrho} dV.$$

Hence requiring $\dot{\mathcal{F}}_{\rm K}(\hat{\mathcal{P}}(t))|_{t=0}$ to vanish along any rigid flow and for every $\mathcal{P}$ amounts to require that the tensor

$$\nabla \varrho \otimes \frac{\partial \sigma_{\rm K}}{\partial \nabla \varrho}$$

be symmetric, thus proving the symmetry of $\mathbf{T}_{\rm K}$.

### E. Balances of forces and torques

A second-grade material can in general convey internal torques by means of a couple stress deriving from the hyperstress [37,38] (see also Sec. 94 of [35]). We show now that the couple stress associated with the hyperstress $\mathsf{M}$ in Eq. (31) vanishes identically. To this end, we consider again a rigid virtual flow like Eq. (32). Since along it the left-hand side of Eq. (5) vanishes, so must also do its right-hand side, provided that the balance equation (28) and the traction laws (29) and (30) are satisfied.

By inserting Eqs. (29) and (30) in Eq. (3) evaluated along flow (32), we readily obtain that

$$\mathcal{W}^{(e)}(\mathcal{P}) = \boldsymbol{v}(o)\cdot\left[\int_{\mathcal{P}}\boldsymbol{b}dV + \int_{\partial\mathcal{P}}(\mathbf{T}_K\boldsymbol{\nu}+\boldsymbol{t}_K)dA\right] + \mathbf{W}\cdot\left[\int_{\mathcal{P}}\boldsymbol{b}\otimes\boldsymbol{x}dV + \int_{\partial\mathcal{P}}(\mathbf{T}_K\boldsymbol{\nu}\otimes\boldsymbol{x}+\boldsymbol{t}_K\otimes\boldsymbol{x}+\boldsymbol{m}\otimes\boldsymbol{\nu})dA\right],$$

where we have introduced the *Korteweg traction*

$$\boldsymbol{t}_K := \nabla_s\left(\varrho^2\frac{\partial\sigma_K}{\partial\nabla\varrho}\cdot\boldsymbol{\nu}\right) - \varrho^2\left(\frac{\partial\sigma_K}{\partial\nabla\varrho}\cdot\boldsymbol{\nu}\right)(\mathrm{div}_s\,\boldsymbol{\nu})\boldsymbol{\nu}. \quad (33)$$

$\mathcal{W}^{(e)}(\mathcal{P})$ vanishes identically for all choices of $\boldsymbol{v}(o)$ and $\mathbf{W}$ if, and only if,

$$\int_{\mathcal{P}}\boldsymbol{b}dV + \int_{\partial\mathcal{P}}(\mathbf{T}_K\boldsymbol{\nu}+\boldsymbol{t}_K)dA = \mathbf{0} \quad \forall\,\mathcal{P}\subset\mathcal{B} \quad (34)$$

and

$$\int_{\mathcal{P}}\boldsymbol{x}\times\boldsymbol{b}dV + \int_{\partial\mathcal{P}}[\boldsymbol{x}\times(\mathbf{T}_K\boldsymbol{\nu}+\boldsymbol{t}_K)+\boldsymbol{\nu}\times\boldsymbol{m}]dA = \mathbf{0} \quad \forall\,\mathcal{P}\subset\mathcal{B}. \quad (35)$$

These equations have a transparent mechanical interpretation; the former represents the balance of all forces acting on $\mathcal{P}$ and the latter represents the balance of all torques exerted by both forces and couples. By applying the divergence theorem, use of Eq. (28) reduces Eq. (34) to

$$\int_{\partial\mathcal{P}}\boldsymbol{t}_K dA = \mathbf{0} \quad \forall\,\mathcal{P}\subset\mathcal{B}, \quad (36)$$

while Eq. (30) and the symmetry of $\mathbf{T}_K$ reduce Eq. (35) to

$$\int_{\partial\mathcal{P}}\boldsymbol{x}\times\boldsymbol{t}_K dA = \mathbf{0} \quad \forall\,\mathcal{P}\subset\mathcal{B}. \quad (37)$$

This latter equation shows that, by its specific structure, the hypertraction $\boldsymbol{m}$ in Eq. (30) does not convey torque, and so the couple stress associated with the hyperstress $\mathsf{M}$ in Eq. (31) vanishes identically.

We now prove directly that both Eqs. (36) and (37) are identically satisfied as a consequence of Eq. (33), as they should, having been obtained by applying the principle of virtual power to a specific virtual flow, whereas both the balance equation (28) and the traction laws (29) and (30) were established by that very principle in its full generality.

Let $\boldsymbol{e}$ be any given unit vector. Then, by Eq. (33), Eq. (36) is equivalent to

$$\int_{\partial\mathcal{P}}\left[\boldsymbol{e}\cdot\nabla_s\left(\varrho^2\frac{\partial\sigma_K}{\partial\nabla\varrho}\cdot\boldsymbol{\nu}\right) - \varrho^2\left(\frac{\partial\sigma_K}{\partial\nabla\varrho}\cdot\boldsymbol{\nu}\right)\mathrm{div}_s\,\boldsymbol{\nu}(\boldsymbol{e}\cdot\boldsymbol{\nu})\right]dA = 0,$$

which, since $\nabla\boldsymbol{e}\equiv\mathbf{0}$, can also be written as

$$\int_{\partial\mathcal{P}}\left\{\mathrm{div}_s\left[\left(\varrho^2\frac{\partial\sigma_K}{\partial\nabla\varrho}\cdot\boldsymbol{\nu}\right)\boldsymbol{e}\right] - \varrho^2\left(\frac{\partial\sigma_K}{\partial\nabla\varrho}\cdot\boldsymbol{\nu}\right)\mathrm{div}_s\,\boldsymbol{\nu}(\boldsymbol{e}\cdot\boldsymbol{\nu})\right\}dA = 0. \quad (38)$$

By applying to Eq. (38) the surface-divergence theorem, we conclude that this equation is identically satisfied for all $\boldsymbol{e}\in\mathbb{S}^2$ and $\mathcal{P}\subset\mathcal{B}$.

We find it convenient rephrasing Eq. (37) in Cartesian components:

$$\int_{\partial\mathcal{P}}\varepsilon_{ijk}[x_j\chi_{;k} - x_j\nu_{h;h}\chi\nu_k]dA = 0, \quad (39)$$

where

$$\chi := \varrho^2\frac{\partial\sigma_K}{\partial\varrho_{,i}}\nu_i,$$

$\varepsilon_{ijk}$ is Ricci's alternator, and a semicolon denotes surface differentiation. Integration by parts and use of the surface gradient-integral theorem in Eq. (25) allow us to rewrite the left-hand side of Eq. (39) as follows:

$$\int_{\partial\mathcal{P}}\varepsilon_{ijk}[x_j\chi\nu_k\nu_{h;h} - \chi x_{j;k} - x_j\nu_{h;h}\chi\nu_k]dA = -\int_{\partial\mathcal{P}}\varepsilon_{ijk}\chi P_{jk}dA, \quad (40)$$

where $P_{jk}$ are the Cartesian components of the projection $\mathbf{P}(\boldsymbol{\nu})$ in Eq. (19). Since $\mathbf{P}(\boldsymbol{\nu})$ is symmetric, the integral on the right-hand side of Eq. (40) vanishes, and so Eq. (37) is identically satisfied for all $\mathcal{P}$.

We thus conclude that the Korteweg traction $\boldsymbol{t}_K$ defined in Eq. (33) represents a system of self-equilibrated contact forces, which, in particular, would not affect the motion of any submerged rigid body. Contrariwise, in general, the hypertraction $\boldsymbol{m}$ in Eq. (30) is not self-equilibrated. However, according to Eqs. (3) and (4), the power $\boldsymbol{m}$ expands against a rigid motion vanishes identically, as, by Eq. (32),

$$\boldsymbol{m}\cdot\frac{\partial\boldsymbol{v}_R}{\partial\nu} = \boldsymbol{m}\cdot\mathbf{W}\boldsymbol{\nu} = -\varrho^2\left(\frac{\partial\sigma_K}{\partial\nabla\varrho}\cdot\boldsymbol{\nu}\right)\boldsymbol{\nu}\cdot\mathbf{W}\boldsymbol{\nu} = 0,$$

since $\mathbf{W}$ is a skew tensor.

The foregoing discussion on the equilibrium of Korteweg fluids served the purpose of identifying Korteweg stress, traction, and hypertraction. Our main interest here lies with the dissipative dynamics of nematic liquid crystals. To derive the basic equations of motion for a dissipative anisotropic Korteweg fluid that describes the acoustic behavior of nematic liquid crystals, we may replace the principle of virtual power with the dissipation principle posited in [42]. In the following section, our treatment of both inertial and viscous forces will follow the pattern of the nematodynamic theory presented in [42], the only substantial difference being the Korteweg forces and torques introduced above.

## III. NEMATOACOUSTIC THEORY

We base our nematoacoustic theory on the postulation that at sufficiently high frequencies a nematic liquid crystal behaves like a special anisotropic Korteweg fluid, symmetric about the local director $\boldsymbol{n}$. The behavior at longer time scale remains the one already described in all textbooks (see, for example, [12,13]); in the presence of a fast phenomenon, such as the propagation of an ultrasonic wave, what survives at the longer time scales is the average of whatever fast variable bears a mechanical meaning. We imagine to distinguish a fast and a slow dynamics, the former evolving as if the latter were not, this latter being influenced only by the time average of the other.

In the fast dynamics, a nematic liquid crystal may reveal features that do not generally characterize its slow dynamics. For example, the very possibility of sound propagation in liquid crystals resides in their being compressible, a property which is generally denied to the slow dynamics. Fast and slow dynamics mutually interfere with one another: the fast dynamics interferes with the slow dynamics by providing time-averaged sources; the slow dynamics in turn drives the background against which the fast dynamics is taking place. Such an interplay will in particular be illuminated by the propagation of ultrasonic waves: they produce an acoustic torque on the nematic director, which later affects the slow director dynamics; this will eventually alter the wave propagation and with it the acoustic torque. Bridging rigorously the different time scales of fast and slow dynamics for ultrasonic wave propagation in nematic liquid crystals is the primary object of this work. We begin by considering the Rayleigh dissipation function for a compressible nematic liquid crystal.

### A. Acoustic dissipation function

At the acoustic time scale, a nematic liquid crystal is regarded as being compressible, and so the velocity field $\boldsymbol{v}$ is no longer solenoidal, though its time average is so. This point of view is not unprecedented in the literature: for example, in the hydrodynamic theory of liquid crystals proposed in [43,44], liquid crystals are compressible fluids. The *acoustic* dissipation function $R_{\mathrm{a}}$, like the dissipation function $R$ in the incompressible limit (see Eq. (63) of [42]), depends on the director $\boldsymbol{n}$, its corotational time derivative

$$\mathring{\boldsymbol{n}} := \dot{\boldsymbol{n}} - \mathbf{W}\boldsymbol{n}, \tag{41}$$

where

$$\mathbf{W} := \frac{1}{2}[(\nabla\boldsymbol{v}) - (\nabla\boldsymbol{v})^{\mathsf{T}}]$$

is the *vorticity* tensor, and the *stretching* tensor

$$\mathbf{D} := \frac{1}{2}[(\nabla\boldsymbol{v}) + (\nabla\boldsymbol{v})^{\mathsf{T}}].$$

$R_{\mathrm{a}}$ is quadratic in the pair $(\mathring{\boldsymbol{n}},\mathbf{D})$ and may also depend on tr $\mathbf{D}$, the new invariant introduced by removing the constraint div $\boldsymbol{v}=0$. Only two quadratic terms in $\mathbf{D}$ containing tr $\mathbf{D}$ may be added to $R$, namely, $(\mathrm{tr}\,\mathbf{D})^2$ and $(\mathrm{tr}\,\mathbf{D})\boldsymbol{n}\cdot\mathbf{D}\boldsymbol{n}$. Therefore, $R_{\mathrm{a}}$ is defined as

$$R_{\mathrm{a}} := \frac{1}{2}\gamma_1\mathring{\boldsymbol{n}}\cdot\mathring{\boldsymbol{n}} + \gamma_2\mathring{\boldsymbol{n}}\cdot\mathbf{D}\boldsymbol{n} + \frac{1}{2}\gamma_3\mathbf{D}\boldsymbol{n}\cdot\mathbf{D}\boldsymbol{n} + \frac{1}{2}\gamma_4\mathbf{D}\cdot\mathbf{D} + \frac{1}{2}\gamma_5(\boldsymbol{n}\cdot\mathbf{D}\boldsymbol{n})^2 + \frac{1}{2}\gamma_6(\mathrm{tr}\,\mathbf{D})^2 + \gamma_7(\mathrm{tr}\,\mathbf{D})\boldsymbol{n}\cdot\mathbf{D}\boldsymbol{n}, \tag{42}$$

where $\gamma_1,\ldots,\gamma_5$ are viscosities coefficients, considered as functions of the mass density $\varrho$.

The dissipation function $R_{\mathrm{a}}$ must be positive semidefinite in all admissible motions. For given $\boldsymbol{n}$, $\mathring{\boldsymbol{n}}$ is only subject to the condition of being orthogonal to $\boldsymbol{n}$, while $\mathbf{D}$ is here an arbitrary symmetric tensor. With no loss in generality, $\mathring{\boldsymbol{n}}$ and $\mathbf{D}$ may be written as

$$\mathring{\boldsymbol{n}} = N\boldsymbol{e}_2 \quad\text{and}\quad \mathbf{D} = \sum_{i,j=1}^{3} A_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j \quad\text{with}\quad A_{ij}=A_{ji}, \tag{43}$$

where $(\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3)$ is an orthonormal frame such that $\boldsymbol{n}=\boldsymbol{e}_1$. By inserting Eq. (43) into Eq. (42), we transform $R_{\mathrm{a}}$ into the sum of four quadratic forms in the independent variables $A_{13}$, $A_{23}$, $(N,A_{12})$, and $(A_{11},A_{22},A_{33})$, respectively:

$$\begin{aligned}R_{\mathrm{a}} &= \left(\frac{1}{2}\gamma_3+\gamma_4\right)A_{13}^2 + \gamma_4A_{23}^2 + \frac{1}{2}\gamma_1N^2 + \gamma_2NA_{12}\\ &\quad + \left(\frac{1}{2}\gamma_3+\gamma_4\right)A_{12}^2 + \frac{1}{2}(\gamma_3+\gamma_4+\gamma_5+\gamma_6+2\gamma_7)A_{11}^2\\ &\quad + (\gamma_6+\gamma_7)A_{11}A_{22} + \frac{1}{2}(\gamma_4+\gamma_6)A_{22}^2 + \gamma_6A_{22}A_{33}\\ &\quad + (\gamma_6+\gamma_7)A_{11}A_{33} + \frac{1}{2}(\gamma_4+\gamma_6)A_{33}^2.\end{aligned}$$

The necessary and sufficient conditions for $R_{\mathrm{a}}$ to be positive semidefinite are the inequalities

$$\gamma_4 \geq 0, \quad \gamma_3 + 2\gamma_4 \geq 0 \tag{44}$$

and the positive semidefiniteness of the symmetric matrices

$$\mathbb{H}_1 := \begin{bmatrix}\gamma_1 & \gamma_2\\ \gamma_2 & \gamma_3+2\gamma_4\end{bmatrix}$$

and

$$\mathbb{H}_2 := \begin{bmatrix}\gamma_3+\gamma_4+\gamma_5+\gamma_6+2\gamma_7 & \gamma_6+\gamma_7 & \gamma_6+\gamma_7\\ \gamma_6+\gamma_7 & \gamma_4+\gamma_6 & \gamma_6\\ \gamma_6+\gamma_7 & \gamma_6 & \gamma_4+\gamma_6\end{bmatrix}.$$

We recall that both $\mathbb{H}_1$ and $\mathbb{H}_2$ are positive semidefinite whenever all their principal minors are non-negative (see, for example, p. 7 of [45] for this positive semidefiniteness criterion) [46]. The principal minors of $\mathbb{H}_1$ are its determinant and the entries $\gamma_1$ and $\gamma_3+2\gamma_4$, and so $\mathbb{H}_1$ is positive semidefinite whenever

$$\gamma_1 \geq 0, \quad \gamma_3+2\gamma_4 \geq 0, \quad\text{and}\quad \gamma_1\gamma_3 + 2\gamma_1\gamma_4 - \gamma_2^2 \geq 0. \tag{45}$$

Clearly, $(45)_2$ reproduces $(44)_2$, which will henceforth be redundant. To ensure that $\mathbb{H}_2$ is positive semidefinite, we begin by requiring that all its leading principal minors are non-negative [47]:

$$\gamma_3+\gamma_4+\gamma_5+\gamma_6+2\gamma_7 \geq 0, \tag{46a}$$

$$\gamma_3\gamma_4+\gamma_3\gamma_6+\gamma_4^2+2\gamma_4\gamma_6+\gamma_4\gamma_5+\gamma_5\gamma_6+2\gamma_4\gamma_7-\gamma_7^2 \geq 0, \tag{46b}$$

$$\gamma_4[\gamma_3\gamma_4+2\gamma_3\gamma_6+\gamma_4^2+3\gamma_4\gamma_6+\gamma_4\gamma_5+2\gamma_5\gamma_6+2\gamma_4\gamma_7-2\gamma_7^2] \geq 0. \tag{46c}$$

It is easily seen, also with the aid of $(44)_1$, that (46c) implies (46b). Three extra inequalities are derived by also requiring the remaining principal minors of $\mathbb{H}_2$ to be non-negative: the only one among these independent from all others is

$$\gamma_4(\gamma_4+2\gamma_6) \geq 0.$$

In summary, $R_a$ is positive semidefinite whenever

$$\gamma_1 \geq 0, \tag{47a}$$

$$\gamma_3+2\gamma_4 \geq 0, \tag{47b}$$

$$\gamma_4 \geq 0, \tag{47c}$$

$$\gamma_4+2\gamma_6 \geq 0, \tag{47d}$$

$$\gamma_3+\gamma_4+\gamma_5+\gamma_6+2\gamma_7 \geq 0, \tag{47e}$$

$$\gamma_1\gamma_3+2\gamma_1\gamma_4-\gamma_2^2 \geq 0, \tag{47f}$$

$$\gamma_4(\gamma_3+\gamma_4+\gamma_5+\gamma_6+2\gamma_7)+2[\gamma_6(\gamma_3+\gamma_4+\gamma_5)-\gamma_7^2] \geq 0. \tag{47g}$$

### B. Nematoacoustic equations

Here we derive the equations that govern acoustic propagation in nematic liquid crystals, assuming that a nematic liquid crystal, as seen from an acoustic wave propagating through it, behaves like an anisotropic, compressible Korteweg fluid with Rayleigh dissipation function $R_a$ as in Eq. (42). More specifically, we assume that at the acoustic time scale (comparable with the wave period) the nematic director $\boldsymbol{n}$ is immobile, so that its dynamics can only be appreciated over much longer time scales. Similarly, we assume that at the acoustic length scale (comparable with the wavelength) $\boldsymbol{n}$ is undistorted, that is, $\nabla\boldsymbol{n}\equiv 0$, so that nematic distortions can only appear over much larger length scales. In particular, this latter assumption implies that the nematoacoustic equations may be derived by taking the elastic energy density $W_e$ as vanishing identically. At the acoustic length scale, the role of $W_e$ is played by the Korteweg energy density $\sigma_K$ introduced in Sec. II above.

#### *1. Balance laws*

Following the general theory presented in [42], in the absence of body forces, the balance of linear momentum is expressed by the equation

$$\varrho\dot{\boldsymbol{v}} = \operatorname{div}(\mathbf{T}_K+\mathbf{T}_{dis}), \tag{48}$$

where the Korteweg stress $\mathbf{T}_K$ is defined as in Eq. (16) and the dissipative stress $\mathbf{T}_{dis}$ is given by

$$\mathbf{T}_{dis} = \frac{1}{2}\left(\boldsymbol{n}\otimes\frac{\partial R_a}{\partial\mathring{\boldsymbol{n}}}-\frac{\partial R_a}{\partial\mathring{\boldsymbol{n}}}\otimes\boldsymbol{n}\right)+\frac{\partial R_a}{\partial\mathbf{D}}. \tag{49}$$

In Eq. (48), the Ericksen stress tensor $\mathbf{T}_E$ defined by

$$\mathbf{T}_E := -(\nabla\boldsymbol{n})^{\mathsf{T}}\frac{\partial W_e}{\partial\nabla\boldsymbol{n}}$$

vanishes identically, as $\nabla\boldsymbol{n}\equiv\mathbf{0}$. It is worth noting that by Eq. (42)

$$\frac{\partial R_a}{\partial\mathring{\boldsymbol{n}}} = \frac{\partial R}{\partial\mathring{\boldsymbol{n}}},$$

whereas

$$\frac{\partial R_a}{\partial\mathbf{D}} = \frac{\partial R}{\partial\mathbf{D}}+[\gamma_6\operatorname{tr}\mathbf{D}+\gamma_7(\boldsymbol{n}\cdot\mathbf{D}\boldsymbol{n})]\mathbf{I}+\gamma_7(\operatorname{tr}\mathbf{D})\boldsymbol{n}\otimes\boldsymbol{n},$$

where $R$ is the nematic dissipation in the incompressible limit.

Similarly, the balance of torques is expressed by the equation [42]

$$\boldsymbol{n}\times\left(-\varrho\frac{\partial\sigma_K}{\partial\boldsymbol{n}}-\frac{\partial R_a}{\partial\mathring{\boldsymbol{n}}}\right) = \mathbf{0}, \tag{50}$$

where again no elastic torque is present, because $\partial W_e/\partial\boldsymbol{n}\equiv\mathbf{0}$. At the acoustic time and length scales, Eq. (50) does not govern the director evolution: as shown below, its time average over an acoustic period will provide the acoustic torque unbalance, responsible for linking the fast acoustic dynamics with the slow director relaxation.

As usual, balance equations (48) and (50) are to be supplemented by the mass continuity Eq. (10). As explained in Sec. II D, the total traction $\boldsymbol{t}$ transmitted through a surface $\mathcal{S}$ within the fluid is given by

$$\boldsymbol{t} = (\mathbf{T}_K+\mathbf{T}_{dis})\boldsymbol{\nu}+\boldsymbol{t}_K,$$

where $\boldsymbol{t}_K$ is as in Eq. (33) and $\boldsymbol{\nu}$ is the outer normal to $\mathcal{S}$. Likewise, the hypertraction $\boldsymbol{m}$ is given by Eq. (30).

#### *2. Constitutive assumption*

Here we write $\sigma_K$ as a specific function of $\varrho$, $\nabla\varrho$, and $\boldsymbol{n}$,

$$\sigma_K(\varrho,\nabla\varrho,\boldsymbol{n}) := \sigma_0(\varrho)+\frac{1}{2}[u_1|\nabla\varrho|^2+u_2(\nabla\varrho\cdot\boldsymbol{n})^2], \tag{51}$$

where the *acoustic susceptibilities* $u_1$ and $u_2$ are assumed to be constitutive parameters independent of $\varrho$ [48]. Clearly, the Korteweg behavior of a nematic liquid crystal at the acoustic time and length scales is anisotropic about $\boldsymbol{n}$. In Eq. (51), the terms in square brackets represent the most general addition to $\sigma_0$ which depends on $\boldsymbol{n}$ and it is both quadratic in $\nabla\varrho$ and frame indifferent. It is easily seen that for such an additional energy to be positive semidefinite, it is necessary and sufficient that $u_1$ and $u_2$ obey the inequalities

$$u_1 \geq 0 \quad \text{and} \quad u_1 + u_2 \geq 0. \tag{52}$$

By Eqs. (16) and (17), the associated Korteweg stress tensor $\mathbf{T}_K$ is then

$$\mathbf{T}_K = -p_K\mathbf{I} - \varrho[u_1 \nabla \varrho \otimes \nabla \varrho + u_2(\nabla \varrho \cdot \boldsymbol{n}) \nabla \varrho \otimes \boldsymbol{n}], \tag{53}$$

where

$$p_K = p_0(\varrho) - \varrho \operatorname{div}\{\varrho[u_1 \nabla \varrho + u_2(\nabla \varrho \cdot \boldsymbol{n})\boldsymbol{n}]\} \tag{54}$$

with

$$p_0(\varrho) := \varrho^2 \frac{d\sigma_0}{d\varrho} \tag{55}$$

an *increasing* function of $\varrho$. For completeness, we record here the form given by Eqs. (30) and (33) to the hypertraction $\boldsymbol{m}$ and to the Korteweg traction $\boldsymbol{t}_K$, respectively, under the constitutive assumption (51):

$$\boldsymbol{m} = -\varrho^2[u_1(\nabla \varrho \cdot \boldsymbol{\nu}) + u_2(\nabla \varrho \cdot \boldsymbol{n})\boldsymbol{n} \cdot \boldsymbol{\nu}]\boldsymbol{\nu},$$

$$\boldsymbol{t}_K = \nabla_s\{\varrho^2[u_1(\nabla \varrho \cdot \boldsymbol{\nu}) + u_2(\nabla \varrho \cdot \boldsymbol{n})\boldsymbol{n} \cdot \boldsymbol{\nu}]\} - \varrho^2[u_1(\nabla \varrho \cdot \boldsymbol{\nu}) + u_2(\nabla \varrho \cdot \boldsymbol{n})\boldsymbol{n} \cdot \boldsymbol{\nu}](\operatorname{div}_s \boldsymbol{\nu})\boldsymbol{\nu}.$$

Finally, it follows from Eqs. (42) and (49) that

$$\begin{aligned}\mathbf{T}_{\rm dis} = {}& \frac{1}{2}\gamma_1(\boldsymbol{n} \otimes \mathring{\boldsymbol{n}} - \mathring{\boldsymbol{n}} \otimes \boldsymbol{n}) + \frac{1}{2}\gamma_2(\boldsymbol{n} \otimes \mathbf{D}\boldsymbol{n} - \mathbf{D}\boldsymbol{n} \otimes \boldsymbol{n}) \\ & + \frac{1}{2}\gamma_2(\mathring{\boldsymbol{n}} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \mathring{\boldsymbol{n}}) + \frac{1}{2}\gamma_3(\boldsymbol{n} \otimes \mathbf{D}\boldsymbol{n} + \mathbf{D}\boldsymbol{n} \otimes \boldsymbol{n}) \\ & + \gamma_4\mathbf{D} + (\gamma_5\boldsymbol{n} \cdot \mathbf{D}\boldsymbol{n} + \gamma_7 \operatorname{tr} \mathbf{D})\boldsymbol{n} \otimes \boldsymbol{n} \\ & + (\gamma_6 \operatorname{tr} \mathbf{D} + \gamma_7\boldsymbol{n} \cdot \mathbf{D}\boldsymbol{n})\mathbf{I}. \end{aligned} \tag{56}$$

In the following subsection, we shall seek plane-wave solutions to Eq. (48) with $\mathbf{T}_K$ and $\mathbf{T}_{\rm dis}$ given as in Eqs. (53) and (56).

### C. Propagation equations

We imagine that an acoustic plane wave is being forced in the fluid by the vibration of a rigid plane at the angular frequency $\omega$, which produces a disturbance in $\varrho$ represented as

$$\varrho = \varrho_0(1 + s). \tag{57}$$

The *condensation* $s$ is given the form

$$s(x,t) = s_0\Re(e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}), \tag{58}$$

where $\Re$ denotes the real part of a complex number, $\boldsymbol{x} := x - o$ with $o$ a given origin, $s_0$ is a small dimensionless parameter, and $\boldsymbol{k}$ is the *complex* wave vector to be determined in terms of $\omega$. Correspondingly, the velocity field $\boldsymbol{v}$ is taken as

$$\boldsymbol{v}(x,t) = s_0\Re(e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)})\boldsymbol{a}, \tag{59}$$

where the amplitude $\boldsymbol{a}$ is an unknown complex vector.

Our program is now seeking solutions in the forms (58) and (59) to the continuity equation (10) and the balance equation of linear momentum (48), under the assumption that only linear terms in the perturbation parameter $s_0$ are to be retained.

To this end, we set

$$E := e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \tag{60}$$

for brevity, and we compute

$$\mathbf{D} = \frac{1}{2}s_0 iE(\boldsymbol{a} \otimes \boldsymbol{k} + \boldsymbol{k} \otimes \boldsymbol{a}) \tag{61}$$

and

$$\mathbf{W} = \frac{1}{2}s_0 iE(\boldsymbol{a} \otimes \boldsymbol{k} - \boldsymbol{k} \otimes \boldsymbol{a}), \tag{62}$$

with the proviso that only their real parts bear a physical meaning. Up to first order in $s_0$, Eq. (10) becomes

$$\omega = \boldsymbol{a} \cdot \boldsymbol{k}. \tag{63}$$

Likewise, also by Eqs. (59) and (61),

$$\varrho\dot{\boldsymbol{v}} = -s_0 i\varrho_0\omega E\boldsymbol{a} + o(s_0).$$

Our postulation here is that at the acoustic time scale $\dot{\boldsymbol{n}} \equiv \boldsymbol{0}$, as $\boldsymbol{n}$ is thought of as being immobile; thus, by Eq. (41), the corotational time derivative $\mathring{\boldsymbol{n}}$ reduces to $\mathring{\boldsymbol{n}} = -\mathbf{W}\boldsymbol{n}$ and the dissipative stress tensor $\mathbf{T}_{\rm dis}$ becomes

$$\begin{aligned}\mathbf{T}_{\rm dis} = {}& \frac{1}{2}(\gamma_1 - \gamma_2)\mathbf{W}\boldsymbol{n} \otimes \boldsymbol{n} - \frac{1}{2}(\gamma_1 + \gamma_2)\boldsymbol{n} \otimes \mathbf{W}\boldsymbol{n} \\ & + \frac{1}{2}(\gamma_2 + \gamma_3)\boldsymbol{n} \otimes \mathbf{D}\boldsymbol{n} + \frac{1}{2}(\gamma_3 - \gamma_2)\mathbf{D}\boldsymbol{n} \otimes \boldsymbol{n} + \gamma_4\mathbf{D} \\ & + (\gamma_5\boldsymbol{n} \cdot \mathbf{D}\boldsymbol{n} + \gamma_7 \operatorname{tr} \mathbf{D})\boldsymbol{n} \otimes \boldsymbol{n} + (\gamma_6 \operatorname{tr} \mathbf{D} + \gamma_7\boldsymbol{n} \cdot \mathbf{D}\boldsymbol{n})\mathbf{I}. \end{aligned} \tag{64}$$

By use of Eqs. (61) and (62) in Eq. (64), we readily arrive at

$$\begin{aligned}\operatorname{div} \mathbf{T}_{\rm dis} = -\frac{1}{2}s_0 E\Bigg\{ & \left[\frac{1}{2}(\gamma_1 - 2\gamma_2 + \gamma_3)(\boldsymbol{k} \cdot \boldsymbol{n})^2 + \gamma_4 k^2\right]\boldsymbol{a} \\ & + \left[\frac{1}{2}(\gamma_3 - \gamma_1 + 4\gamma_7)(\boldsymbol{a} \cdot \boldsymbol{n})(\boldsymbol{k} \cdot \boldsymbol{n}) + (\gamma_4 + 2\gamma_6)(\boldsymbol{a} \cdot \boldsymbol{k})\right]\boldsymbol{k} \\ & + \left[\frac{1}{2}(\gamma_3 - \gamma_1 + 4\gamma_7)(\boldsymbol{k} \cdot \boldsymbol{n})(\boldsymbol{k} \cdot \boldsymbol{a}) + \frac{1}{2}(\gamma_1 + 2\gamma_2 + \gamma_3)(\boldsymbol{a} \cdot \boldsymbol{n})k^2 + 2\gamma_5(\boldsymbol{a} \cdot \boldsymbol{n})(\boldsymbol{k} \cdot \boldsymbol{n})^2\right]\boldsymbol{n}\Bigg\}. \end{aligned} \tag{65}$$

On the other hand, by Eqs. (53) and (54), we show that

$$\operatorname{div} \mathbf{T}_K = -\nabla p_K + o(s_0) = -s_0\varrho_0 iE\{c_0^2 + \varrho_0^2[u_1 k^2 + u_2(\boldsymbol{k} \cdot \boldsymbol{n})^2]\}\boldsymbol{k} + o(s_0), \tag{66}$$

where

$$c_0 := \sqrt{\frac{dp_0(\varrho_0)}{d\varrho_0}}, \tag{67}$$

see Eq. (55), is the velocity of sound in the isotropic compressible fluid described by Eq. (51) with $u_1=u_2=0$. Up to the first order in $s_0$, the balance equation of linear momentum (48) then reduces to the purely kinematic form

$$\begin{aligned}2i\omega\boldsymbol{a} = {} & 2i\{c_0^2+\varrho_0^2[u_1k^2+u_2(\boldsymbol{k}\cdot\boldsymbol{n})^2]\}\boldsymbol{k} + \left[\frac{1}{2}(\nu_1-2\nu_2+\nu_3)\right.\\ & \left.\times(\boldsymbol{k}\cdot\boldsymbol{n})^2+\nu_4k^2\right]\boldsymbol{a} + \left[\frac{1}{2}(\nu_3-\nu_1+4\nu_7)(\boldsymbol{a}\cdot\boldsymbol{n})(\boldsymbol{k}\cdot\boldsymbol{n})\right.\\ & \left.+(\nu_4+2\nu_6)(\boldsymbol{a}\cdot\boldsymbol{k})\right]\boldsymbol{k} + \left[\frac{1}{2}(\nu_3-\nu_1+4\nu_7)(\boldsymbol{k}\cdot\boldsymbol{n})(\boldsymbol{k}\cdot\boldsymbol{a})\right.\\ & \left.+\frac{1}{2}(\nu_1+2\nu_2+\nu_3)(\boldsymbol{a}\cdot\boldsymbol{n})k^2+2\nu_5(\boldsymbol{a}\cdot\boldsymbol{n})(\boldsymbol{k}\cdot\boldsymbol{n})^2\right]\boldsymbol{n},\end{aligned} \tag{68}$$

where we have set

$$\nu_i := \frac{\gamma_i}{\varrho_0} \quad \text{for} \quad i=1,\ldots,7 \tag{69}$$

and all $\gamma_i$'s are evaluated at the unperturbed density $\varrho_0$.

Equation (68) must be supplemented with the mass continuity equation (63). We let $\boldsymbol{k}$ and $\boldsymbol{a}$ be represented as

$$\boldsymbol{k}=k\boldsymbol{e} \quad \text{and} \quad \boldsymbol{a}=a_e\boldsymbol{e}+a_n\boldsymbol{n}, \tag{70}$$

with $\boldsymbol{e}\in\mathbb{S}^2$ designating the propagation direction and $k$, $a_e$, and $a_n$ all complex numbers to be determined. In particular, we set

$$k=k_1+ik_2.$$

The imaginary part $k_2$ of $k$ will be associated with the *attenuation* of the wave: when $k_2>0$, its reciprocal represents the length over which the wave amplitude is reduced by the factor $1/e$; such a length is also called the *attenuation length*.

We write Eq. (63) in the form

$$ka_e+ka_n\cos\beta=\omega, \quad \text{with} \quad \cos\beta := \boldsymbol{e}\cdot\boldsymbol{n}. \tag{71}$$

It follows from Eqs. (70) and (71) that, whenever $\sin\beta=0$, $a_e$ and $a_n$ are not uniquely defined; we resolve this ambiguity by setting $a_n=0$ for $\sin\beta=0$.

Before solving Eqs. (68) and (71), we introduce new dimensionless variables defined as

$$k' := \frac{c_0}{\omega}k =: \left(\frac{c_0}{c}+ik_2'\right), \quad a_e' := \frac{a_e}{c_0}, \quad a_n' := \frac{a_n}{c_0},$$
$$\text{and} \quad \nu_i' := \frac{\omega}{c_0^2}\nu_i \text{ for } i=1,\ldots,7, \tag{72}$$

where we have set

$$k_1 =: \frac{\omega}{c}, \tag{73}$$

with $c$ the velocity of sound in the nematic medium still to be determined. Written in the new variables, Eq. (71) readily delivers

$$a_e' = \frac{1}{k'} - a_n'\cos\beta. \tag{74}$$

Similarly, by Eq. (63), taking the inner product of both sides of Eq. (68) with $\boldsymbol{k}$, we obtain the scalar equation

$$\begin{aligned}2i = {} & 2i\left(1+\frac{1}{4}\omega^2\tau^2k'^2\right)k'^2 + \left[\frac{1}{2}(\nu_1'-2\nu_2'+\nu_3')\cos^2\beta\right.\\ & +\frac{1}{2}(\nu_3'-\nu_1'+4\nu_7')(\cos\beta+a_n'k'\sin^2\beta)\cos\beta\\ & \left.+2(\nu_4'+\nu_6')\right]k'^2 + \left[\frac{1}{2}(\nu_3'-\nu_1'+4\nu_7')\cos\beta\right.\\ & +\frac{1}{2}(\nu_1'+2\nu_2'+\nu_3')(\cos\beta+a_n'k'\sin^2\beta)\\ & \left.+2\nu_5'(\cos\beta+a_n'k'\sin^2\beta)\cos^2\beta\right]k'^2\cos\beta,\end{aligned} \tag{75}$$

where use has been made of Eq. (74) and $\tau$ is the *anisotropic* characteristic time defined by

$$\tau^2 := 4\frac{\varrho_0^2}{c_0^4}(u_1+u_2\cos^2\beta). \tag{76}$$

Moreover, taking the inner product of both sides of Eq. (68) with $\boldsymbol{n}$ and using again Eq. (74), we arrive at

$$\begin{aligned}& 2i(\cos\beta+a_n'k'\sin^2\beta)\\ & = 2i\left(1+\frac{1}{4}\omega^2\tau^2k'^2\right)k'^2\cos\beta\\ & \quad+\left[\frac{1}{2}(\nu_1'-2\nu_2'+\nu_3')\right.\\ & \quad\left.\times\cos^2\beta+\nu_4'\right](\cos\beta+a_n'k'\sin^2\beta)k'^2\\ & \quad+\left[\frac{1}{2}(\nu_3'-\nu_1'+4\nu_7')(\cos\beta+a_n'k'\sin^2\beta)\cos\beta\right.\\ & \quad\left.+(\nu_4'+2\nu_6')\right]k'^2\cos\beta+\left[\frac{1}{2}(\nu_3'-\nu_1'+4\nu_7')\cos\beta\right.\\ & \quad+\frac{1}{2}(\nu_1'+2\nu_2'+\nu_3')(\cos\beta+a_n'k'\sin^2\beta)\\ & \quad\left.+2\nu_5'(\cos\beta+a_n'k'\sin^2\beta)\cos^2\beta\right]k'^2.\end{aligned} \tag{77}$$

Equations (75) and (77) are algebraic in $k'$ and $a_n'$: they determine all propagation modes allowed by this theory. We begin by considering special instances of these equations, which are easier to solve. We first set $\sin\beta=0$, so that $a_n'=0$ and the wave is longitudinal. Then Eq. (75) becomes

$$i=i\left(1+\frac{1}{4}\omega^2\tau^2k'^2\right)k'^2+(\nu_3'+\nu_4'+\nu_5'+\nu_6'+2\nu_7')k'^2, \tag{78}$$

and Eq. (77) reduces to the same Eq. (78) with both sides multiplied by $\cos\beta$. It also follows from Eq. (74) that

$$a_e'=\frac{1}{k'}. \tag{79}$$

We now set $\cos\beta=0$. Thus Eq. (74) implies that $a_e'$ is still given by Eq. (79). Moreover, Eqs. (75) and (77) become

$$i=i\left(1+\frac{1}{4}\omega^2\tau^2k'^2\right)k'^2+(\nu_4'+\nu_6')k'^2 \tag{80}$$

and

$$a_n'k'\left[2i-\frac{1}{2}(\nu_1'+2\nu_2'+\nu_3'+2\nu_4')k'^2\right]=0,$$

whence it follows that $a_n'=0$. Thus the wave propagating at right angles with the nematic director is also longitudinal.

Though both Eqs. (78) and (80) can easily be solved explicitly, their solutions are given by rather cumbersome expressions, which do not make their interpretation transparent. We rather prefer studying the limit of small viscosities, where all $\nu_i'$ are treated as perturbation parameters of one and the same order. To this end, we first consider the inviscid limit, in which all viscosities are set equal to zero. Equation (75) then becomes

$$\left(1+\frac{1}{4}\omega^2\tau^2k'^2\right)k'^2-1=0, \tag{81}$$

which, together with Eq. (77), also requires $a_n'=0$, and correspondingly implies that $a_e'$ is as in Eq. (79). The only solution of Eq. (81) with positive real part has $k_2'=0$ and

$$\frac{c}{c_0}=\frac{\omega\tau}{\sqrt{2(\sqrt{1+\omega^2\tau^2}-1)}}. \tag{82}$$

We drop the solution with negative real part, as it represents the same wave propagating in the opposite direction. We also drop the other two purely imaginary solutions, as they do not represent traveling waves. Since $\tau$ depends on $\beta$, the dispersion described by Eq. (82) is anisotropic. Thus, as expected, in the inviscid limit the wave is not attenuated and, as also shown in Fig. 1, $c\geq c_0$, for all $\omega\geq 0$, being $c=c_0$ only for $u_1=u_2=0$. Moreover, asymptotically

$$c\approx\frac{c_0}{\sqrt{2}}\sqrt{\omega\tau}\ \text{ for }\ \omega\tau\gg 1.$$

We now assume that all dimensionless viscosities $\nu_i'$ are $O(s_0)$ and continue in the viscosities the solution to the propagation equations already found in the inviscid limit. In particular, we write $k'$ as

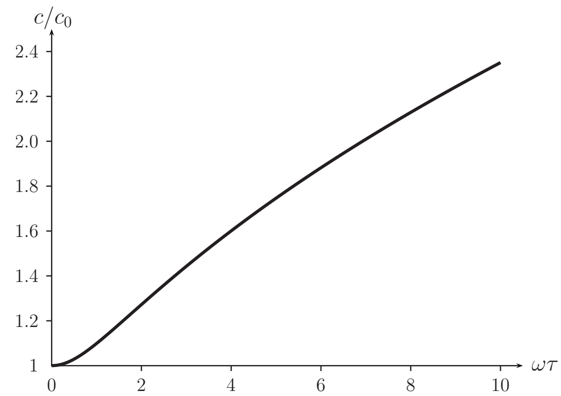


FIG. 1. The speed of sound $c$ in the Korteweg fluid described by Eq. (51) scaled to the speed $c_0$ corresponding to the limit of zero acoustic susceptibilities, $u_1=u_2=0$.

$$k'=\frac{c_0}{c}+h_1'+ik_2' \tag{83}$$

and we assume that both $h_1'$ and $k_2'$ are $O(s_0)$. By Eq. (72), assuming $k_2'\ll 1$ amounts to assume that the attenuation length of the propagating wave is much smaller that the wavelength. Intuitively, this is grounded in the assumption that all viscosities are small, in the sense made precise by requiring that $\nu_i'\ll 1$, for all $i$.

Inserting Eq. (83) into Eqs. (75), (77), and (74), at the lowest order of approximation in $s_0$, we obtain that

$$h_1'=0, \tag{84a}$$

$$k_2'=\frac{1}{2\dfrac{c}{c_0}+\omega^2\tau^2\dfrac{c_0}{c}}[\nu_4'+\nu_6'+(\nu_3'+2\nu_7')\cos^2\beta+\nu_5'\cos^4\beta], \tag{84b}$$

$$a_n'=-i\frac{1}{2}\left(\frac{c_0}{c}\right)\frac{\cos\beta}{\sin^2\beta}[\nu_2'+\nu_3'+2\nu_7'-(\nu_2'+\nu_3'-2\nu_5'+2\nu_7')\cos^2\beta-2\nu_5'\cos^4\beta]\ \text{ for }\ \sin\beta\neq 0, \tag{84c}$$

$$a_e'=\frac{c}{c_0}-i\left(\frac{c}{c_0}\right)^2k_2'-a_n'\cos\beta, \tag{84d}$$

where $c$ is expressed by Eq. (82) as a function of both $\omega$ and $\beta$.

The solutions to Eqs. (75) and (77) for which the real part vanishes in the inviscid limit can also be continued as all dimensionless viscosities $\nu_i'$ move away from zero. The continued solution with positive real part of $k'$ can be represented as

$$k'=h_1'+ih_2',$$

where

$$h_2' = \frac{\sqrt{2(1\sqrt{1+\omega^2\tau^2})}}{\omega\tau}$$

and $h_1' = O(s_0)$. It turns out that at the lowest order of approximation

$$\frac{h_2'}{h_1'} = \frac{2\sqrt{1+\omega^2\tau^2}}{\nu_4' + \nu_6' + (\nu_3' + 2\nu_7')\cos^2\beta + \nu_5'\cos^4\beta} = O\left(\frac{1}{s_0}\right).$$

This would thus correspond to a wave propagating with a speed $c$ much larger that $c_0$ and with an attenuation length much shorter than the wavelength. Such a wave could not indeed propagate, and so it will hereafter be disregarded, though it might rise and compete with the wave that propagates in the asymptotic limit of small viscosities in the complete nonlinear analysis of propagation equations (75) and (77).

#### *1. Anisotropic dispersion*

The graph in Fig. 1 fails to represent the anisotropy in the speed of sound. To capture this feature of $c$, we define the *relative* sound speed *anisotropy* $\Delta c$ as

$$\Delta c := \frac{c - c|_{\beta=\pi/2}}{c|_{\beta=0}}. \tag{85}$$

$\Delta c$ is a function of both $\omega$ and $\beta$, which vanishes for $\beta = \frac{\pi}{2}$. To distinguish in $\Delta c$ the dependence on $\omega$ from the dependence on $\beta$, we find it convenient letting

$$\varepsilon := \frac{u_2}{u_1} \tag{86}$$

and assuming that $\varepsilon$ is a small parameter. Then, by Eq. (82), Eq. (85) yields

$$\Delta c = \varepsilon^2 f(\omega\tau_1)\cos^2\beta + O(\varepsilon^4), \tag{87}$$

where $\tau_1$ is defined by

$$\tau_1^2 := 4\frac{u_1\varrho_0^2}{c_0^4}, \tag{88}$$

and

$$f(x) := \frac{1}{4}\frac{x^2 - 2(\sqrt{1+x^2}-1)}{\sqrt{1+x^2}(\sqrt{1+x^2}-1)}. \tag{89}$$

It readily follows from Eq. (89) that

$$f(x) = \frac{1}{8}x^2 + O(x^4) \ \text{ for } \ x \ll 1, \ \text{ and } \ \lim_{x\to\infty} f(x) = \frac{1}{4}.$$

As shown by Fig. 2, $f$ is a positive, strictly increasing function, so that, in particular, the speed of propagation along the nematic director is larger than the speed of propagation at right angles to it. The prediction in Eq. (87) also agrees with the observations of [23] for *p*-*n*-butyl-aniline (MBBA) at 21 °C and wave frequency 10 MHz under the action of an aligning magnetic field with strength 5 Oe. The data for $\Delta c$ were represented in Fig. 2 of [23] as $\Delta c = A\cos^2\beta$, with $A = 12.5\times 10^{-4}$.

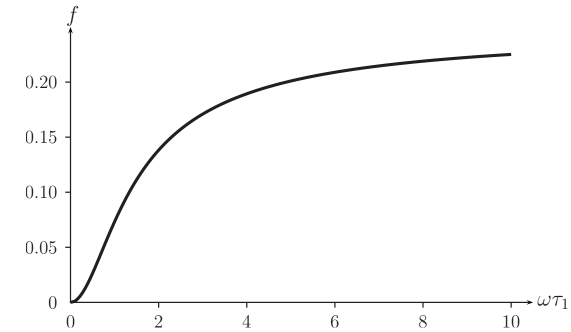


FIG. 2. In the limit where the susceptibility $u_2$ is much smaller than the susceptibility $u_1$, the frequency dependence of the speed anisotropy $\Delta c$ in Eq. (85) is represented by the function $f$ in Eq. (89), here plotted against $\omega\tau_1$. At small frequencies, $f$ is quadratic; at large frequencies, it saturates to $\frac{1}{4}$.

#### *2. Anisotropic attenuation*

Here we shall further explore the dependence of the wave attenuation $k_2$ on the propagation direction. By Eqs. (69) and (72), we readily derive from Eq. (84b) the dimensional form of the attenuation:

$$k_2 = \frac{\omega^2}{2\varrho_0 c_0^3}\frac{1}{\frac{c}{c_0} + \frac{1}{2}\omega^2\tau^2\frac{c_0}{c}}[\gamma_4 + \gamma_6 + (\gamma_3 + 2\gamma_7)\cos^2\beta + \gamma_5\cos^4\beta]. \tag{90}$$

It is worth noting that $k_2 \geq 0$ for both $\beta=0$ and $\beta = \frac{\pi}{2}$, as a consequence of inequalities (47c) and (47e). It would be desirable to prove that $k_2 \geq 0$ also for all $\beta \in [0,\pi]$, but this does not seem to be an immediate consequence of inequalities (47).

The angular dependence exhibited by Eq. (90) coincides with that predicted by Lee and Eringen [49] in their theory for wave propagation in nematic liquid crystals phrased within the general *micromorphic* theory of continuum mechanics first put forward by Eringen and Suhubi [50] and later extended by Eringen [51–53]. However, as pointed out in [54,55], at the lowest order in the condensation, this theory does not predict dispersion of sound, and consequently the frequency dependence of the attenuation is classically quadratic. In particular, it is shown in [55] that this is indeed a feature common both to the theories presented in [43,44] and to the theory of Leslie [56,57]. While the dependence on $\beta$ of $k_2$ in Eq. (90) has been widely confirmed [54,58], a purely quadratic dependence of $k_2$ on $\omega$ has no experimental ground [54,55]. Since in our theory $c$ depends on $\omega$ and $\tau$ does not vanish, Eq. (90) exhibits indeed a nonquadratic dependence on $\omega$, which we now explore more closely, introducing an appropriate measure of attenuation anisotropy.

The attenuation *anisotropy* $\Delta k_2$ is here defined as

$$\Delta k_2 := k_2 - k_2|_{\beta=0}, \tag{91}$$

which like $\Delta c$ is a function of both $\omega$ and $\beta$. By assuming again that $\varepsilon$ in Eq. (86) is a small parameter, we easily give Eq. (91) the following form:

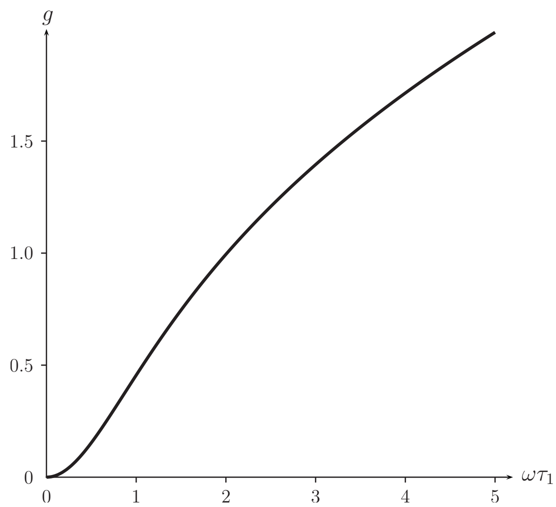


FIG. 3. In the limit where the susceptibility $u_2$ is much smaller than the susceptibility $u_1$, the frequency dependence of the attenuation anisotropy $\Delta k_2$ in Eq. (91) is represented by the function $g$ in Eq. (93), here plotted against $\omega\tau_1$. At small frequencies, $g$ is quadratic; at large frequencies, it exhibits a square-root growth.

$$\Delta k_2 = \frac{c_0\gamma}{4\sqrt{2}\varrho_0^3 u_1} g(\omega\tau_1)G(\beta) + O(\varepsilon^2), \tag{92}$$

where

$$g(x) := \frac{x\sqrt{\sqrt{1+x^2}-1}}{\sqrt{1+x^2}} \tag{93}$$

and

$$G(\beta) := -\sin^2\beta\left(1 + \frac{\gamma_5}{\gamma}\cos^2\beta\right), \tag{94}$$

with

$$\gamma := \gamma_3 + \gamma_5 + 2\gamma_7. \tag{95}$$

The function $g$, which is plotted in Fig. 3, possesses the following asymptotic behaviors:

$$g(x) = \frac{1}{\sqrt{2}}x^2 + O(x^4) \quad \text{for} \quad x \ll 1$$

and

$$g(x) = \sqrt{x} + O\left(\frac{1}{\sqrt{x}}\right) \quad \text{for} \quad x \gg 1.$$

For $\gamma > 0$, by Eqs. (91), (92), and (94), $g$ is proportional to the difference between the attenuations in the propagation parallel to $\boldsymbol{n}$ and in the propagation orthogonal to $\boldsymbol{n}$. Since $g \geq 0$, Eq. (94) shows in particular that for $\gamma > 0$ the attenuation in the orthogonal propagation is smaller than the attenuation in the parallel propagation. It is to be noted how the graph of $g$ differs from the classical parabolic form, characteristic of the case where dispersion is absent, which in the present setting would correspond to the limit of zero acoustic susceptibilities, $u_1 = u_2 = 0$. The early measurements of Lord and Labes [4] for $\Delta k_2$ at $\beta = \frac{\pi}{2}$ and for various frequencies, shown in their Fig. 2, reproduce qualitatively the behavior of $g$.

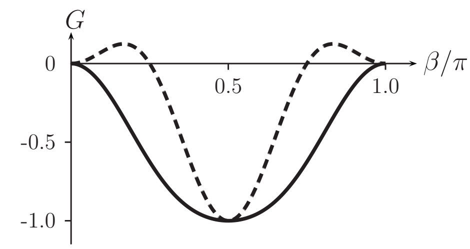


FIG. 4. In the limit where the susceptibility $u_2$ is much smaller than the susceptibility $u_1$, the angular dependence of the attenuation anisotropy $\Delta k_2$ in Eq. (91), for $\gamma > 0$, is represented by the function $G$ in Eq. (94), here plotted against $\beta$ for $\gamma_5 = \frac{1}{2}\gamma$ (solid line) and $\gamma_5 = -2\gamma$ (dashed line).

In Fig. 4, we illustrate the graphs of $G$ against $\beta$ in two exemplary cases, for $\gamma_5 = \frac{1}{2}\gamma$ and $\gamma_5 = -2\gamma$. For $\gamma > 0$, the former graph reproduces the qualitative features shown by the graph in Fig. 1 of [4], which fits the experimental values of $\Delta k_2$ measured at a given frequency for various propagation angles. For $\gamma > 0$ and $\gamma_5 \geq 0$, $\Delta k_2$ is negative for all propagation angles, and so the less attenuated wave travels orthogonally to the nematic director; clearly, when either $\gamma$ or $\gamma_5$ is negative, this conclusion is not necessarily valid.

### 3. Acoustic intensity

The *acoustic intensity* carried by the wave is defined as

$$I_a := \langle p_K \boldsymbol{v} \cdot \boldsymbol{e} \rangle, \tag{96}$$

where $\langle \cdot \rangle$ denotes time average over a period and $\boldsymbol{e} \in \mathbb{S}^2$ designates the direction of propagation. To within the second order in $s_0$, by Eqs. (54), (73), and (96),

$$I_a = \varrho_0 s_0^2 \left[ c_0^2 c + (u_1 + u_2\cos^2\beta)\varrho_0^2 \frac{\omega^2}{c} \right] \langle (\Re E)^2 \rangle$$
$$= I_0 \left[ \left(\frac{c}{c_0}\right) + \frac{1}{4}\omega^2\tau^2 \left(\frac{c_0}{c}\right) \right], \tag{97}$$

where

$$I_0 := \frac{1}{2}\varrho_0 s_0^2 c_0^3 e^{-2k_2 \boldsymbol{x}\cdot\boldsymbol{e}} \tag{98}$$

is the acoustic intensity of the wave in the limit of no acoustic susceptibility and $c$ is given by Eq. (82) as a function of $\omega$ and $\beta$.

The graph of $I_a$ scaled to $I_0$ is shown in Fig. 5 against $\omega\tau$; it reveals how acoustic susceptibility increases acoustic intensity. One should keep in mind that two independent sources of anisotropy are hidden in $I_a$, namely, $\tau$ and $k_2$.

### 4. Acoustic torque

As already pointed out, one major issue related to ultrasonic wave propagation in nematic liquid crystals is to explain the ability of ultrasound to act on the nematic director [59]. The Korteweg nature of nematic liquid crystals at the

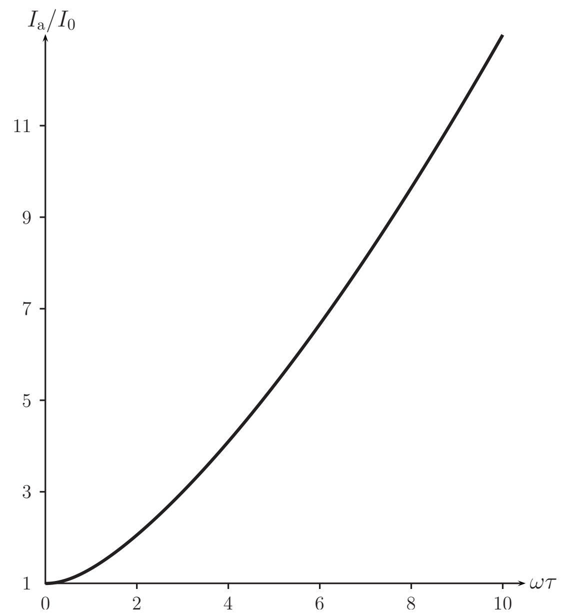


FIG. 5. The acoustic intensity $I_a$ for the Korteweg fluid described by Eq. (51) scaled to the acoustic intensity $I_0$ corresponding to the limit of zero acoustic susceptibilities, $u_1=u_2=0$.

time and length scales characteristic of ultrasonic propagation has been here the main idea to explain the acoustic interaction with the molecular alignment.

We read through the balance equation of torques (50) the action exerted by the acoustic field on the nematic director. It follows from Eqs. (51) and (42) that

$$\frac{\partial \sigma_K}{\partial \boldsymbol{n}} = u_2[(\nabla \varrho \cdot \boldsymbol{n}) \nabla \varrho - (\nabla \varrho \cdot \boldsymbol{n})^2 \boldsymbol{n}] \tag{99}$$

and

$$\left\langle \frac{\partial R_a}{\partial \mathring{\boldsymbol{n}}} \right\rangle = \gamma_1 \mathring{\boldsymbol{n}} + \gamma_2[\mathbf{D}\boldsymbol{n} - (\boldsymbol{n} \cdot \mathbf{D}\boldsymbol{n})\boldsymbol{n}]. \tag{100}$$

In particular, for the acoustic flow considered here, where the director does not librate, $\dot{\boldsymbol{n}} \equiv 0$ and Eq. (100) becomes

$$\frac{\partial R_a}{\partial \mathring{\boldsymbol{n}}} = -\gamma_1 \mathbf{W} + \gamma_2[\mathbf{D}\boldsymbol{n} - (\boldsymbol{n} \cdot \mathbf{D}\boldsymbol{n})\boldsymbol{n}],$$

which, by Eqs. (60)–(62), implies that

$$\left\langle \frac{\partial R_a}{\partial \mathring{\boldsymbol{n}}} \right\rangle = \mathbf{0}.$$

Thus, at time scales longer than the acoustic period, Eq. (50) reveals an unbalanced acoustic torque $\boldsymbol{K}_a$ which has its origin in the Korteweg coupling we have postulated; $\boldsymbol{K}_a$ is defined as

$$\boldsymbol{K}_a := -\boldsymbol{n} \times \left\langle \varrho \frac{\partial \sigma_K}{\partial \boldsymbol{n}} \right\rangle = -u_2 \boldsymbol{n} \times \langle \varrho \nabla \varrho \otimes \nabla \varrho \rangle \boldsymbol{n}. \tag{101}$$

For $\varrho$ as in Eq. (57), at the lowest order of approximation, we obtain that

$$\langle \varrho \nabla \varrho \otimes \nabla \varrho \rangle = \frac{1}{4} \mathrm{sgn}(u_2) \frac{I_0}{c_0} \omega^2 \tau_2^2 \left( \frac{c_0}{c} \right)^2 \boldsymbol{e} \otimes \boldsymbol{e}, \tag{102}$$

where sgn denotes the sign function,

$$\tau_2^2 := 4 \frac{\varrho_0^2}{c_0^4} |u_2|, \tag{103}$$

$c$ is as in Eq. (82), and $I_0$ is given by Eq. (98) with $k_2$ as in Eq. (90). By inserting Eq. (102) into Eq. (101), we arrive at

$$\boldsymbol{K}_a = -\mathrm{sgn}(u_2) K_0 (\boldsymbol{n} \cdot \boldsymbol{e}) \boldsymbol{n} \times \boldsymbol{e}, \tag{104}$$

where

$$K_0 := \frac{1}{4} \frac{I_0}{c_0} \omega^2 \tau_2^2 \left( \frac{c_0}{c} \right)^2. \tag{105}$$

A few remarks are suggested by Eq. (104). First, since $K_0 \geq 0$, $\boldsymbol{K}_a$ is an *aligning* torque, that is, it tends to bring $\boldsymbol{n}$ along the propagation direction $\boldsymbol{e}$, only if $u_2 < 0$; if $u_2 > 0$, it is a *misaligning* torque, which tends to make $\boldsymbol{n}$ orthogonal to $\boldsymbol{e}$. The experiments reported by Selinger and co-workers in [26,28–30] appear to confirm that $u_2 > 0$ for the materials they have examined. Second, it may appear that $\boldsymbol{K}_a$ behaves essentially like a magnetic torque, the case with positive diamagnetic anisotropy being the analog of the case with negative acoustic susceptibility $u_2$ and, conversely, the case with negative diamagnetic anisotropy being the analog of the case with positive acoustic susceptibility $u_2$. This analogy, however, is only formal, as the dependence of $K_0$ on the propagation direction makes the dependence of $\boldsymbol{K}_a$ on the angle between $\boldsymbol{n}$ and $\boldsymbol{e}$ more complicated than it appears from Eq. (104). In case of pure acoustic relaxation of the nematic director, such a dependence might result in a relaxation law more complicated than a simple exponential decay.

## IV. CONCLUSION

The nematoacoustic theory presented in this paper is variational in that it retraces the source of the interaction between the acoustic field and the nematic molecular alignment in an elastic coupling of capillary type. It remains a phenomenological theory, as the acoustic susceptibilities $u_1$ and $u_2$ introduced in Eq. (51) need to be determined experimentally by exploring the consequences of the theory. Among these, some appear particularly promising, namely, the anisotropy and dispersion in sound speed and the nonconventional frequency dependence of wave attenuation. These features, which other theories do not possess, stem from the assumed Korteweg nature of the acoustic coupling. Were they confirmed by an assessment of the experimental data surpassing the mere qualitative agreement we could report here, our constitutive assumption on the nature of the acoustic coupling would be more firmly established.

Strictly speaking, our propagation equations in Sec. III C were derived under the assumption that the director $\boldsymbol{n}$ is uniform and immobile, as if it were held fixed by some external action, such as an applied magnetic field. This is indeed the situation envisaged in the wealth of experimental studies recalled above. In the absence of such external causes, the director is free to vary in time and be distorted in space. These variations take place at time and length scales much larger than the acoustic characteristic times and lengths, so that especially an ultrasonic wave propagates locally in an undistorted medium, where our equations still apply. Such a reasoning might suggest that the evolution of the director, which is governed by the complete balance of torques, including the elastic, viscous, and acoustic torques, would interfere with the wave propagation only marginally, by affecting locally its anisotropic character. This would indeed be correct, were the sound speed independent of the propagation direction. On the contrary, we have shown above that this is not the case in our theory. Such an *acoustic birefringence* causes the director texture to alter the ultrasonic propagation: the director, which can be distorted by an acoustic wave, in turn causes the refringence of the distorting wave. Studying the ultrasonic propagation in a moderately distorted nematic medium is a challenge the theory proposed here should next face. One might also learn from it how to steer an acoustic wave by acting on the nematic texture through controllable external actions.

## ACKNOWLEDGMENTS

I am indebted to G. Assanto, G. De Matteis, and A. M. Sonnet for suggestions and discussion about the theory presented in this paper.

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[46] A principal submatrix of a $n \times n$ matrix $\mathbb{M}$ is a submatrix of $\mathbb{M}$ whose principal diagonal is part of the principal diagonal of $\mathbb{M}$. A principal minor of $\mathbb{M}$ is the determinant of a principal submatrix of $\mathbb{M}$.

[47] A leading principal minor of a $n \times n$ matrix $\mathbb{M}$ is the determinant of a principal submatrix of $\mathbb{M}$ identified by the first $j$ rows and the first $j$ columns of $\mathbb{M}$, with $j \leq n$.

[48] It is easily seen that taking $u_1$ and $u_2$ as functions of $\varrho$ would not affect the linearized plane-wave analysis performed in Sec. III C: the acoustic susceptibilities appearing there would simply coincide with their values at the unperturbed density $\varrho_0$. Nevertheless, in general, the Korteweg pressure $p_{\mathrm{K}}$ acquires a more complicated expression when $u_1$ and $u_2$ depend on $\varrho$.

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